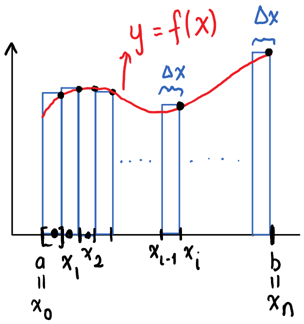
# Lecture 23

Wednesday, November 2, 2016 8:33 AM

The area A of the region S that lies under the graph of the continuous function f, $a \le x \le b$ is the limit of the area of approximating rectangles




$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\Delta x}$$

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

right endpoint of $i^{th}$ subinterval $[x_{i-1}, x_i]$

**Remark** 1) Since the function is continuous, limit always exists.

2) , we get the same value when we consider left endpoints i.e $A = \lim_{n\to\infty} L_n$

Actually instead of using left or right endpoints, we could take the height of the $i^{th}$ rectangle to be the value of f at any number in interval $[x_{i-1}, x_i]$ called the sample point, denoted by $x_i^*$.

$$\underbrace{\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) . \Delta x}}$$

## Definite Integral

If f is a function defined on $[a,b]$, the definite integral of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

b → upper limit

a → lower limit

provided the limit exists.

a,b ≡ limits of integration

f(x) = integrand.

$f(x)$ = integrand.

$\int_a^b f(x)\,dx \equiv$ Definite Integral

$\int f(x)\,dx \equiv$ indefinite Integral
$\equiv$ antiderivative.

Defn A function $f$ is said to be integrable on $[a,b]$ if $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$ exists.

Theorem If $f$ is continuous on $[a,b]$ or if $f$ has only finite number of jump discontinuities, then $f$ is integrable on $[a,b]$ i.e. $\int_a^b f(x)\,dx$ exists.

Remarks

$$\int_a^b f(x)\,dx = \int_a^b f(r)\,dr = \int_a^b f(t)\,dt$$

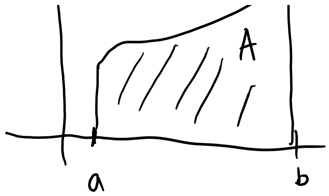
Theorem If $f$ is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\,\Delta x}_{R_n} = A$$

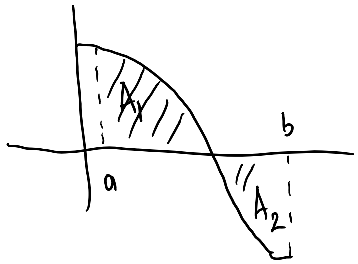
where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$

Remark


$$\int_a^b f(x)\,dx = A$$




a




$$\int_a^b f(x)\,dx = A_1 - A_2$$

$$\boxed{\begin{gathered}\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \\ \Delta x = \frac{b-a}{n} \\ x_i = a + i\Delta x\end{gathered}}$$

Ex $\displaystyle\int_0^2 (1+3x^2)\,dx$

$f(x) = 1 + 3x^2$

$a = 0$, $b = 2$

Since $f$ is continuous, it is integrable.

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}, \quad x_i = a + i\Delta x = 0 + i\cdot\frac{2}{n} = \frac{2i}{n}$$

$$f(x_i) = 1 + 3x_i^2 = 1 + 3\left(\frac{2i}{n}\right)^2 = 1 + 3\cdot\frac{4i^2}{n^2} = 1 + \frac{12i^2}{n^2}$$

$$\int_0^2 (1+3x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(1 + \frac{12i^2}{n^2}\right)}_{f(x_i)} \cdot \underbrace{\frac{2}{n}}_{\Delta x} = 10$$

See notes from Friday.

## Properties of Integrals

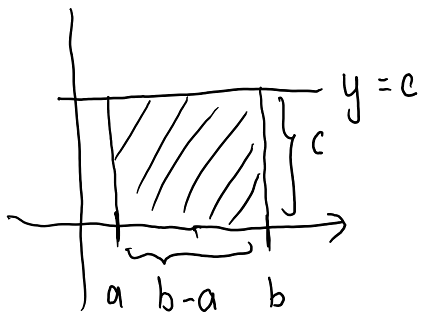
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad (1)$$

a b

$$\int_a^a f(x)\,dx = 0$$ ②

$$\underbrace{\int_a^a f(x)\,dx} = \underbrace{-\int_a^a f(x)\,dx} \Rightarrow \int_a^a f(x)\,dx = 0$$

$$\int_a^b c\,dx = c(b-a), \quad c \equiv \text{cnst}$$

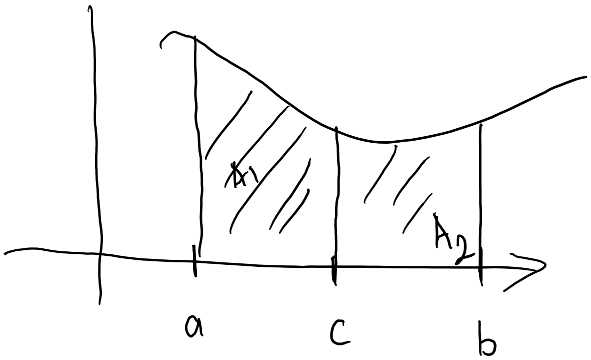



$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

$$A_1 + A_2$$




Ex $\int_0^{10} f(x)\,dx = 7$ , $\int_0^{8} f(x)\,dx = 3$. Find $\int_8^{10} f(x)\,dx$

By above property,

$$\int_0^{8} f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$$

$$3 + \int_8^{10} f(x)\,dx = 7 \Rightarrow \int_8^{10} f(x)\,dx = 4$$

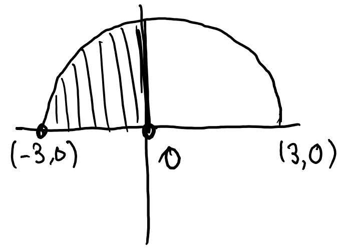
Ex $\int_{-3}^{0} \sqrt{9-x^2}\,dx = ?$

Find the area under the curve $y = {}^{+}\sqrt{9-x^2}$

betn $x = -3$ and $x = 0$ . , $y = \sqrt{9-x^2}$

betn $x = -3$ and $x = 0$.

$y = \sqrt{9-x^2}$




$y = \sqrt{9-x^2}$

$y^2 = 9-x^2 \Rightarrow x^2+y^2 = 9$

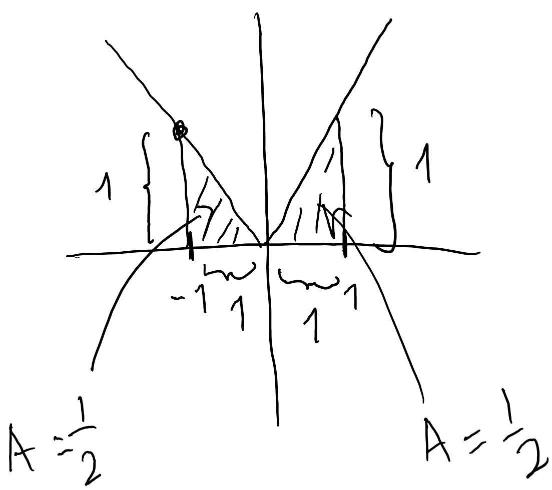
$$\int_{-3}^{0} \sqrt{9-x^2}\,dx = \frac{9\pi}{4} \quad \left(\frac{1}{4} \text{ Area of circle of radius } 3\right)$$

<u>Ex</u> $$\int_{-1}^{1} |x|\,dx = \frac{1}{2} + \frac{1}{2} = 1$$

$\frac{1}{2}.b.h$

$= \frac{1}{2}.1.1$

$= \frac{1}{2}$




$$\int_{-1}^{1} |x|\,dx = \underbrace{\int_{-1}^{0} |x|\,dx}_{\frac{1}{2}} + \underbrace{\int_{0}^{1} |x|\,dx}_{1/2}$$